Workshop on Novel Approaches to Uncertainty
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Linking Probability and Fuzzy Set Theories Using Likelihoods, Membership Functions, and Bayes Theorem

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Different Uncertainties Within A Problem

Suppose we have a reliability problem for a complex system. Reliability is a probability.

Suppose we elicit knowledge about the components and processes of this system from experts. We get an initial (prior) probability estimate of the system's performance.

One of the components has poor reliability, so we seek additional information before expensive testing is proposed.

Suppose the vendor of this component supplies us with membership functions about the performance.

How can we take his membership functions (fuzzy) and incorporate those into the reliability (probability) problem?

Humans Contribute to Uncertainty

- **Formal, structured elicitation of expert knowledge** counters common biases arising from human cognition and behavior.
- Adds rigor, defensibility, and increased ability to update the judgments.
- **Utilizes the way people think, work, and problem solve.**

Mathematical Theories - Frameworks for Expert Thinking

Characteristics

- Set based (crisp or fuzzy)
- Axiomatic
- Calculus (rules for implementing axioms)
- Consistent / coherence
- Computationally practical (??)
- Measure based (not all!)

Goal: Provide Metrics for Uncertainty

For combining uncertainties there needs to be a bridge between the various theories.

Probability: A Calculus for the Uncertainty of Outcomes

A foundation for the theory of probability is:

- A well-defined specification of a set of *outcomes*, and its subsets.
- An adherence to the ***law of the excluded middle***; i.e., any outcome either belongs to a set or does not belong to a set—*Crisp Set*
- A calculus (or algebra) based on some behavioristic axioms, involving numbers between 0 and 1, which can be made operational after *E* is performed.
- The *outcome* of *E* is uncertain.

P(*E*) describes the uncertainty about the *outcome.*

The Three Axioms of the Calculus of Probability

i) $0 \leq P(\mathrm{A}) \leq 1$

ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

iii) $P(A \cap B) = P(A \mid B)\, P(B)$
$= 0$ if $A \cap B$? $\varnothing$;

where $P(A \mid B)$ is the conditional probability of A should B occur and $A \cap B$? $\varnothing$ which implies event A is independent of event B if $P(A \mid B) = P(A)$ and vice versa.

Probability is Coherent

Interpretations of Probability

Theory does NOT tell us how to arrive at or interpret $P(E)$.
At least 11 different theories or interpretations or meanings of probability.

Focus on two with this calculus (coherence)

- Relative Frequency Theory
- Personalistic or Subjective Theory

There is not a unique interpretation of probability

Relative Frequency Theory

FOUNDERS: Aristotle, Venn, von Mises, and Reichenbach

INTERPRETATION:

- Measure of an empirical, objective and physical fact of the external world, independent of human attitudes, opinions, models and simulations.
- Never relative to evidence or opinion.
- Like mass, it is determined by observations on the nature of the real world.
- **Only** known aposteriori, i.e., **only** upon observation.
- Property of a *collective*, i.e., scenarios involving events that repeat again, and again, e.g., games of chance (like coin tossing) and social mass phenomena (like actuarial and insurance problems).
- Excludes one-of-a-kind or individual events.

Personalistic or Subjective Theory

FOUNDERS: Borel, Ramsey, Savage, DeFinetti

INTERPRETATION:

- No such a thing as an objective probability, unknown probability or correct probability
- Degree of belief of a given person at a given time, measured in some sense.
- Degree of belief could be expressed as a willingness to bet. $P\{E\} = p$ => willingness to bet \$*p* in exchange for \$1, should the event occur, and staking \$(1-*p*) in exchange for \$1, should the event not occur. [two sided bet]
- Accounts for all history (prior to observation or settling the bet) including expertise, mathematical modeling, experience, knowledge, records, etc.)
- Includes Bayesian

Likelihood

Fisher: What we can find from a sample is the likelihood of any particular value of [the parameter], if we define the likelihood as the quantity **proportional** to the probability that, from a population having that particular value, the [observed sample] should be obtained.

Probability ? Likelihood

- The variable quantity in likelihood is the hypothesis
- Probability refers to an outcome of the experiment.
- All outcomes have probabilities that sum to 1.0, but that is not necessarily so regarding hypotheses.
- No need for an axiom of summing likelihoods such as for probability.

Likelihood and Probability

Likelihoods Do NOT sum or integrate to 1.0

$$L(\theta \| x) \propto P(x | \theta)$$

| means conditional probability

|| means “given”

Bayes Theorem

Discrete probability form:

$$P(A_j|B) = [P(B|A_j)\ P(A_j)] \,/\, P(B),\ \ j=1,2,\ \ldots$$

where $P(B) = \Sigma_j\, P(B|A_j)\ P(A_j)$.

Continuous form using probability density functions g and f:

$$g(q|x)\ =\ [f(x|q)\ \ g(q)] \,/ \int f(x|q)\ g(q)\ \mathrm{d}q$$

OR $g(q|x)\ =\ [f(x|q)\ \ g(q)] \,/\, f(x)$

OR $g(q|x)\ =\ [L(q\ ||\ x)\ \ g(q)] \,/\, f(x)$

Fuzzy Set Theory: A Calculus for Imprecision

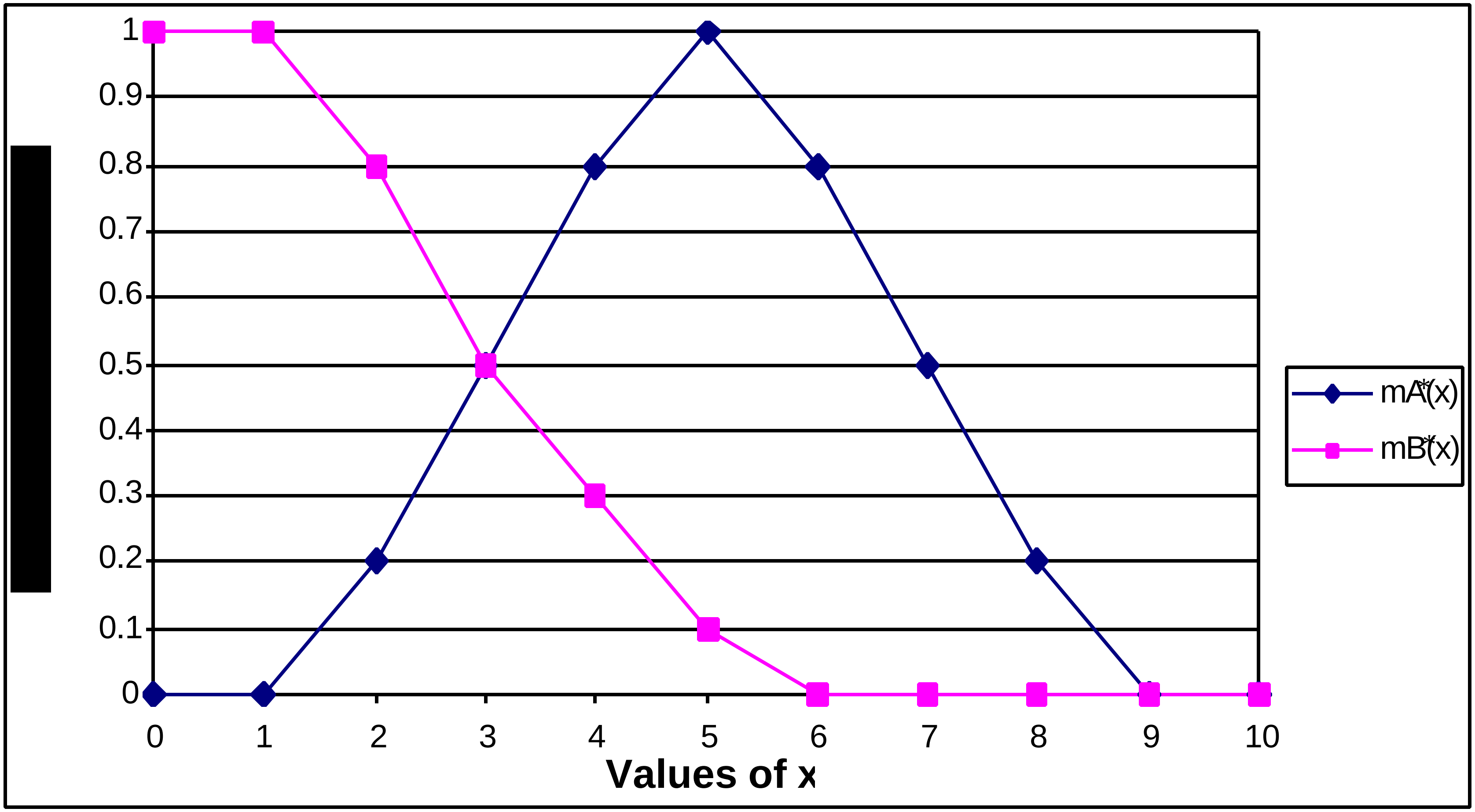
- Introduced by Lotfi Zadeh in 1965.
- A mathematical construct in set theory that enhances classical set theory.
- Useful for quantification: turning rules into numerical functions (and the way people think).
- Designed for capturing a vagueness type of uncertainty.

Fuzzy Set Theory: A Calculus for Imprecision

Consider the set of integers X=**{1, 2, …, 10}.**
Define a subset, of X, where
$A^* = \{x : x \in X$ and x is "medium"$\}$
Defining A* implies a precision in defining what is "medium".

Most agree that 5 is a "medium" integer. What about 3? Is it "medium" or is it "small"? We are uncertain about the classification of 3. Because of this vagueness, we are unable to define the subset.

Membership Functions for Small and Medium



Fuzzy Sets and Membership Functions

Membership functions are a way of dealing with the above vagueness (or uncertainty).

$m_{A^*}(x)$ = membership function of A* reflects the expert's assessment of how **likely** it is that $x \in A^*$.

The expert assigns to each $x \in X$ a number, $m_{A^*}(x)$, and this is done for all subsets of the type that are of interest. The set , A*, is called a **fuzzy set**.

For crisp sets; all $x \in X$, $m_{A^*}(x) = 0$ or 1.

Membership Functions ? Probability

Membership functions: an epistemic uncertainty from the lack of knowledge about how to classify *x*.

P(*A*) can be interpreted as a two-sided bet in the subjective or personalistic probability interpretation, dealing with the uncertainty associated with the outcome of the experiment, an aleatory uncertainty.

Fuzzy sets reject the law of the excluded middle.

Probability adheres to the law of the excluded middle.

$P(A) \in [0,1]$ $\qquad \Sigma_j P(A_j) = 1$

$m_{A^*}(x) \notin [0,1]$ $\qquad \Sigma_j m_{A^*}(x_j)$? 1

A Calculus for Fuzzy Membership Functions

Axioms for combining two or more fuzzy sets:

$m_{A^* \cup B^*}(x) = \max[m_{A^*}(x), m_{B^*}(x)]$
$m_{A^* \cap B^*}(x) = \min[m_{A^*}(x), m_{B^*}(x)]$
$m_{A^{*\prime}}(x) = 1 - m_{A^*}(x)$

If $m_{A^*}(x) = m_{\mathbf{B}^*}(x)$, then $A^* = B^*$
If $m_{A^*}(x) \leq m_{\mathbf{B}^*}(x)$, then $A^* \subseteq B^*$

Linking Fuzzy and Probability

Membership functions and probability are both subjective assessments.

Membership functions are likelihoods.

Bayes Theorem connects likelihood and probability.

Bayes Theorem provides the bridge between fuzzy and probability.

Probability of a Fuzzy Set?

$P(A^*) = P(X \in A^*)$

two types of uncertainty: the outcome of the experiment $X=x$ and the membership of x in A^*.

$$P(A^*) = P(X \in A^*) = \Sigma_x P(X \in A^* | X=x) \cdot P(x)$$

$$P(x \in A^* | \, m_{A^*}(x)) = \Sigma_x P(x \in A^* | \, m_{A^*}(x)) \cdot P(x)$$

Apply Bayes theorem to middle term:

$$P(x \in A^* | \, m_{A^*}(x)) \propto \Sigma_x P(m_{A^*}(x) \,|\, x \in A^*) \cdot P(x)$$

Middle term here is likelihood and the membership function. Combining the last two equations:

$$P(X \in A^*; \, m_{A^*}(x)) \propto \Sigma_x \, m_{A^*}(x) \cdot P(x \in A^*) \cdot P(x)$$

Probability of a Fuzzy Set

Using membership functions, likelihoods and Bayes theorem, we get the probability of a fuzzy set:

$$P(X \in A^*) \propto \Sigma_x \, m_{A^*}(x) \cdot P(x \in A^*) \cdot P(x)$$

Linking fuzzy and probability theories.

Two Theories Linked

General Information Theories provide alternatives to probability for characterizing different kinds of uncertainties within the same problem and all conformity with experts' thinking as part of formal elicitation principles.

Additional linkages between these theories are required.

Mathematical Probability

A set function P defined for all sets in a Boolean field F having these properties is referred to as the probability measure on F :

- For every event, E, in Boolean field, F, there is associated a real non-negative number $P(E)$, called the probability of event E.
- If $E_1, E_2, \ldots$ is a countably infinite sequence of mutually disjoint sets in F whose union is in F then
  - $P((E_i) = \Sigma P(E_i)$
- For sample space, R, $P(R)$=1
- P is the probability measure (or probability distribution) on the Borel field F, $B(F)$

Probability: A Calculus for the Uncertainty of Outcomes

The *outcome* of E is uncertain.

- $P(E)$ describes the uncertainty about the *outcome*.
- The bet is two-sided and it will be unambiguously settled when E is performed, and the *outcome* is observed.
- Thus, $P(E)$ can be interpreted and made operational.
- Note that probability theory does not tell how to arrive upon a $P(E)$, nor in its abstract form even interpret $P(E)$. This is a job of a statistician/analyst.

Uncertainties

Many meanings and connotations to different communities.

Propose a broad definition that includes:

- chance or randomness
- lack of knowledge or imprecise knowledge
- vagueness or ambiguity
- lack of precision (e.g., in measurements)
- approximation and inference (e.g., modeling)

Humans Are NOT Probabilistic Thinkers

- Studies have shown humans do not think well in terms of probability. {Difficult}

- They cannot estimate probability well {Miscalibrated}

- They underestimate uncertainty
- {Over confidence bias}

Probability is not recommended for elicitation

Uncertainty Quantification

Broad Definition — the process of characterizing, estimating, propagating, and analyzing various kinds of uncertainty (including variability) for a complex decision problem.

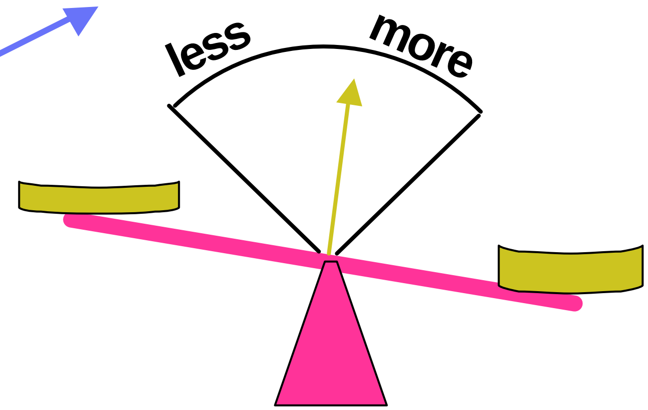
For complex computer and physical models — focuses upon measurement, computational, parameter (including sensitivities of outputs to input values), and modeling uncertainties leading to verification and validation.

Additional Uncertainty: "Human In The Loop"

Sources of uncertainty

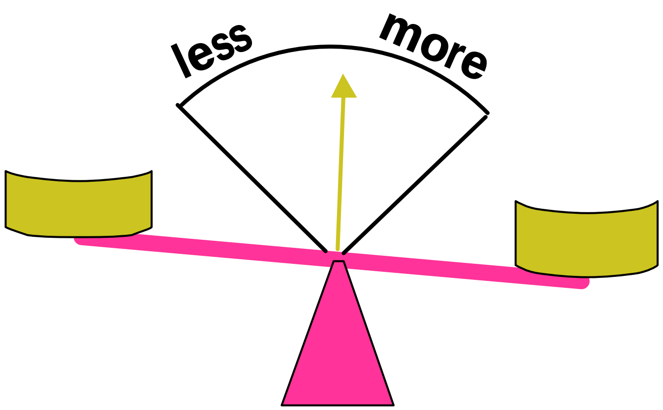
- Measurements
- Mathematical models
- Numerical models
- Surrogate models (statistical)
- Model parameters
- Scenarios

The expert is making decisions about all of these choices and inducing uncertainties in the process.



Formal, Structured Elicitation of Expertise and Expert Judgment

- Minimizes biases
- Provides documentation
- Utilizes the way people think, work, and problem solve
- Provides what is necessary for uncertainty quantification:
  - Sources,
  - Quantification,
  - Estimates and Updates,
  - Methods of propagation



Mathematics (Theories) Handling Ignorance, Ambiguity, Vagueness and the Way People Think

- Probability Theory (different interpretations within e.g., Frequentist, Subjective/Bayesian)
- Possibility Theory (crisp or fuzzy set)
- Fuzzy Sets
- Dempster-Schafer (Evidence)Theory
- Choquet Capacities
- Upper and Lower Probabilities
- Convex Sets
- Interval Analysis Theories
- Information Gap Decision Theory (non measure based)